FACTORIZABLE RIBBON QUANTUM GROUPS IN LOGARITHMIC CONFORMAL FIELD THEORIES

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ABSTRACT. We review the properties of quantum groups occurring as Kazhdan–Lusztig dual to logarithmic conformal field theory models. These quantum groups at even roots of unity are not quasitriangular but are factorizable and have a ribbon structure; the modular group representation on their center coincides with the representation on generalized characters of the chiral algebra in logarithmic conformal field models.

1. INTRODUCTION

The relation of quantum groups to conformal field theory, discussed since [1, 2, 3, 4], has been formulated in the context of vertex-operator algebras as the Kazhdan–Lusztig correspondence [5]. In a very broad sense (and very roughly), it states that whenever “something occurs” in the representation category of a vertex-operator algebra, “something similar occurs” in the representation category of an appropriate quantum group; in other words, there is a functor relating these two categories, although this functor does not have to be either left- or right-exact. In this broad sense, the Kazhdan–Lusztig correspondence is therefore a *principle* rather than a precise statement; the details of the functor have to be worked out in each particular case. For rational conformal field theories, a certain complication follows from the fact that the chosen vertex-operator-algebra representation category is semisimple, while the quantum-group one is not, and additional “semisimplification” (taking the quotient over tilting modules) is needed to ensure the equivalence [6]. But in logarithmic conformal field theories [7, 8], the representation category is already nonsemisimple on the conformal field theory side, and therefore no quotients need to be a priori taken on the quantum group side. The Kazhdan–Lusztig correspondence extended to the logarithmic realm shows remarkable properties and, in particular, extends to modular-group representations [9, 10, 11].

A more “physical” point of view on the Kazhdan–Lusztig correspondence originates from the observation that screening operators that commute with a given vertex-operator algebra generate a quantum group and, moreover, the vertex-operator algebra and the quantum group are characterized by being each other’s commutant,

$$[\text{vertex-operator algebra}, \text{quantum group}] = 0,$$

with each of the objects in this relation allowing reconstruction of the other. But this picture also applies more as a principle than as a precise statement, and therefore needs

a clarification as well. First, the screenings proper generate only the upper-triangular subalgebra of the quantum group in question; the entire quantum group has to be reconstructed either by introducing contour-removal operators (see [12] and the references therein) or, somewhat more formally, by taking Drinfeld's double [9, 13]. Second, in seeking the commutant of a quantum group, it must be specified *where* it is sought, i.e., what free-field operators are considered (in particular, what are the allowed momenta of vertex operators or whether vertex operators are allowed at all; cf., e.g., [14, 15] in the nonlogarithmic case).

For several logarithmic conformal field theories, the Kazhdan–Lusztig correspondence has been shown to have very nice properties [9, 10, 13, 11], being "improved" compared to the rational case. Somewhat heuristically, such an "improvement" may relate to the fact that the field content in a logarithmic model is determined not by the cohomology but by the kernel of the screening(s) (more precisely, by the kernel of a differential constructed from the screenings; we recall that the rational models are just the cohomology of such a differential, cf. [16, 17]). Most remarkably, the Kazhdan–Lusztig correspondence extends to modular group representations. We recall that a modular group representation in a logarithmic conformal field model is generated from the characters $(\chi_a(\tau))$ of the model by $\mathcal{T}$- and $\mathcal{S}$-transformations, the latter being expressed as

$$\chi_a(-\frac{1}{\tau}) = \sum_b S_{ab}\chi_b(\tau) + \sum_{b'} S'_{ab'}\psi_{b'}(\tau), \tag{1.1}$$

which involves certain functions $\psi_{b'}$, which are not characters [18, 19, 20, 13], with

$$\psi_{a'}(-\frac{1}{\tau}) = \sum_b S'_{a'b}\chi_b(\tau) + \sum_{b'} S'_{a'b'}\psi_{b'}(\tau) \tag{1.2}$$

(the χ and ψ together can be called generalized, or extended characters, for the lack of a better name). On the other hand, in quantum-group terms, the general theory in [21] (also see [22, 23]), which has been developed in an entirely different context, can be adapted to the quantum groups that are dual to logarithmic conformal field theories, with the result that a modular group representation is indeed defined on the quantum group center. This representation turns out to be *equivalent* to the representation generated from the characters.

Another instance where logarithmic conformal field theories and the corresponding ("dual") quantum groups show similarity is the fusion (Verlinde) algebra/Grothendieck ring. The existing data suggest that the Grothendieck ring of the Kazhdan–Lusztig-dual quantum group coincides with or "is closely related to" the fusion of the chiral algebra representations on the conformal field theory side. Two remarks are in order here: first, comparing a Grothendieck ring with a fusion algebra implies that the latter is understood "in a K_0-version," when all indecomposable representations are perforce replaced with

direct sums (cf. a discussion of this point in [19]);[1] second, when the logarithmic conformal field theory has a rational subtheory, the representations of this rational theory are to be excluded from the comparison (this is not unnatural though, cf. [26]).

The quantum groups that have so far occurred as dual to logarithmic conformal field theories are a quantum $s\ell(2)$ and a somewhat more complicated quantum group, a quotient of the product of two quantum $s\ell(2)$. They are dual to logarithmic conformal field theories in the respective classes of $(p, 1)$ and (p, p') models. In either case, the Kazhdan–Lusztig-dual quantum group is at an *even* root of unity. In either case, the quantum group has a set of crucial properties, which may therefore be conjectured to be common to the quantum groups that are dual to logarithmic conformal models. These properties and the underlying structures are reviewed here. At present, their derivation is only available in a rather down-to-earth manner, by direct calculation, which somewhat obscures the general picture. In what follows, we skip the calculation details and concentrate on the final results and on the interplay of different structures associated with the quantum group.

We thus continue the story as seen from the quantum-group side, following the ideology and results in [9, 10, 13, 11]. The necessary excursions to logarithmic conformal field theory (see [8, 27, 25, 28, 19, 9, 13] and the references therein) are basically limited to what is needed to appreciate the similarities with quantum-group structures. When we need to be specific (which is almost always the case, because we do not claim any generality here), we choose the simplest of the two basic examples, the $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ quantum group dual to the $(p, 1)$ logarithmic conformal field theory models, but we indicate the properties shared by the quantum group $\mathcal{U}_{p,p'}$ dual to the (p, p') logarithmic models wherever possible.

The quantum group dual to the logarithmic $(p, 1)$ model is $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ at an even root of unity

$$\mathfrak{q} = e^{\frac{i\pi}{p}} \tag{1.3}$$

The three generators E, F, and K satisfy the relations

$$\begin{gathered} KEK^{-1} = \mathfrak{q}^2 E, \quad KFK^{-1} = \mathfrak{q}^{-2} F, \\ [E, F] = \frac{K - K^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}} \end{gathered} \tag{1.4}$$

and the “constraints”

$$E^p = F^p = 0, \quad K^{2p} = 1. \tag{1.5}$$

[1] Whenever indecomposable representations are involved, it is of course possible (and more interesting) to consider fusion algebras where indecomposable representations are treated honestly, i.e., are not replaced by the direct sum of their irreducible subquotients [24, 25]. The correspondence with quantum groups may also extend from the “K_0/Grothendieck-style” fusion to this case (also see **3.3.2** below).

We note that Eqs. (1.3)–(1.4) already imply that E^p, F^p, and K^{2p} are central, which then allows imposing (1.5) (but K^p, which is also central, is *not* set equal to unity, which makes the difference with a smaller but more popular version, the so-called *small* quantum $s\ell(2)$). As a result, $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ is $2p^3$-dimensional. The quantum group $\mathcal{U}_{p,p'}$ dual to the (p,p') logarithmic model is $2p^3p'^3$-dimensional. We note that the "constraint" imposed on its Cartan generator is $K^{2pp'}=1$.

The Hopf algebra structure of $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ (comultiplication Δ, counit ϵ, and antipode S) is described by

$$\begin{gathered}\Delta(E)=1\otimes E+E\otimes K,\quad \Delta(F)=K^{-1}\otimes F+F\otimes 1,\quad \Delta(K)=K\otimes K,\\ \epsilon(E)=\epsilon(F)=0,\quad \epsilon(K)=1,\\ S(E)=-EK^{-1},\quad S(F)=-KF,\quad S(K)=K^{-1}.\end{gathered}\tag{1.6}$$

The simplicity of (1.4)–(1.6) is somewhat misleading. This quantum group (as well as $\mathcal{U}_{p,p'}$) has interesting algebraic properties, the *central* role being played by its center.

Quantum group center and structures on it. On the quantum-group side, the main arena of the Kazhdan–Lusztig correspondence is the quantum group center Z. Of course, it contains the ("quantum") Casimir element(s) and the algebra that they generate, but this does not exhaust the center.

The center carries an $SL(2,\mathbb{Z})$ representation, whose definition [21, 22, 23] requires three types of structure: Drinfeld and Radford maps $\boldsymbol{\chi}$ and $\widehat{\boldsymbol{\phi}}$, and a ribbon element $\boldsymbol{v}$. The action of $\mathcal{S}=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\in SL(2,\mathbb{Z})$ on the center is given by

$$\begin{tikzcd} & \mathsf{Ch} \arrow[dl, "\boldsymbol{\chi}"'] \arrow[dr, "\widehat{\boldsymbol{\phi}}"] & \\ \mathsf{Z} & & \mathsf{Z} \arrow[ll, bend left, "\mathcal{S}^{-1}"] \end{tikzcd}\tag{1.7}$$

where Ch is the space of q-characters (linear functionals invariant under the coadjoint action), and the action of $\mathcal{T}=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\in SL(2,\mathbb{Z})$ essentially by (multiplication with) the ribbon element,

$$\mathsf{Z}\xrightarrow{\ \boldsymbol{v}\ }\mathsf{Z}.\tag{1.8}$$

(Our definition of $\widehat{\boldsymbol{\phi}}$ is swapped with its inverse compared to the standard conventions.)

A possible way to look at the center is to first identify a number of central elements associated with traces over irreducible representations and then introduce appropriate pseudotraces. The ("quantum") trace over an irreducible representation gives an element of Ch, i.e., a functional on the quantum group that is invariant under the coadjoint representation; these invariant functionals (q-characters) can then be mapped into central elements. This does *not* cover the entire center. But then projective quantum-group modules yield additional q-characters, obtained by taking traces, informally speaking, of nondiagonal

components of the quantum group action, nondiagonal in terms of the filtration of projective modules. This gives a basis γ_A in the space Ch of q-characters and hence a basis in the center.

Also, the Drinfeld map $\boldsymbol{\chi} : \mathsf{Ch} \to \mathsf{Z}$ is an isomorphism of associative commutative algebras. Therefore, the center contains (an isomorphic image of) the Grothendieck ring of the quantum group, which is thus embedded into a larger associative commutative algebra.[2]

In Sec. 2, we review the construction of the Radford map $\widehat{\boldsymbol{\phi}}$. In Sec. 3, we recall the necessary facts about the (irreducible and projective) representations of the relevant quantum groups; their Grothendieck rings are also discussed there. In Sec. 4, we recall the M[onodromy] "matrix," the Drinfeld map $\boldsymbol{\chi}$, and the ribbon element. Together with the Radford map, these serve to define the modular group action, which we finally consider in Sec. 5.

2. Radford map and related structures

We consider the Radford map $\widehat{\boldsymbol{\phi}} : U^* \to U$; the construction of $\widehat{\boldsymbol{\phi}}$ and its inverse involves a cointegral and an integral.

2.1. Integral and cointegral.

2.1.1. Integral. For a Hopf algebra U, a *right integral* $\boldsymbol{\lambda}$ is a linear functional on U satisfying

$$(\boldsymbol{\lambda} \otimes \mathrm{id})\Delta(x) = \boldsymbol{\lambda}(x)1 \quad \forall x \in U. \tag{2.1}$$

Such a functional exists in a finite-dimensional Hopf algebra and is unique up to multiplication [29].

2.1.2. Remark. The name *integral* for such a $\boldsymbol{\lambda} \in U^*$ is related to the fact that (2.1) is also the property of a right-invariant integral on functions on a group. Indeed, for a function f on a group G, $\Delta(f)$ is the function on $G \times G$ such that $\Delta(f)(x,y) = f(xy)$, $x, y \in G$. Then the invariance property $\int f(?\,y) = \int f(?)$ can be written as $(\int \otimes \mathrm{id})\Delta(f) = \int f$.

2.1.3. Cointegral. The dual object to $\boldsymbol{\lambda}$, an integral for U^*, is sometimes called a cointegral for U. We give it in the form needed below, when it is a two-sided cointegral.[3]

[2] That the center contains the image of the Grothendieck ring but is larger than it has a counterpart in logarithmic conformal field theory, where the set of chiral algebra characters χ_a is to be extended by other functions $\psi_{a'}$ in order to define a modular group action [18, 19, 13] and thus, presumably, to construct the space of torus amplitudes (also see [20]).

[3] U^* is therefore assumed unimodular, which turns out to be the case for the quantum groups considered below.

A two-sided *cointegral* $\boldsymbol{\Lambda}$ is an element in U such that

$$x\boldsymbol{\Lambda} = \boldsymbol{\Lambda}x = \epsilon(x)\boldsymbol{\Lambda} \quad \forall x \in U.$$

Clearly, the cointegral defines an embedding of the trivial representation of U into the regular representation. The normalization $\boldsymbol{\lambda}(\boldsymbol{\Lambda}) = 1$ is typically understood.

2.2. The Radford map. Let U be a Hopf algebra with a right integral $\boldsymbol{\lambda}$ and a two-sided cointegral $\boldsymbol{\Lambda}$. The Radford map $\widehat{\boldsymbol{\phi}} : U^* \to U$ and its inverse $\widehat{\boldsymbol{\phi}}^{-1} : U \to U^*$ are given by[4]

$$\widehat{\boldsymbol{\phi}}(\beta) = \beta(\boldsymbol{\Lambda}')\boldsymbol{\Lambda}'', \quad \widehat{\boldsymbol{\phi}}^{-1}(x) = \boldsymbol{\lambda}(S(x)?). \tag{2.2}$$

2.2.1. Lemma ([29, 31])**.** *$\widehat{\boldsymbol{\phi}}$ and $\widehat{\boldsymbol{\phi}}^{-1}$ are inverse to each other and intertwine the left actions of U on U and U^*, and similarly for the right actions.*

Here, the left-U-module structure on U^* is given by $a \rightharpoonup \beta = \beta(S(a)?)$ (and on U, by the regular action). In particular, restricting to the space of q-characters (see **A.1**) gives

$$\widehat{\boldsymbol{\phi}} : \mathsf{Ch} \to \mathsf{Z}.$$

Proof. We first establish an invariance property of the integral,

$$\boldsymbol{\lambda}(xy')y'' = \boldsymbol{\lambda}(x'y)S^{-1}(x''). \tag{2.3}$$

Indeed, $\boldsymbol{\lambda}(xy')y'' = \boldsymbol{\lambda}(x'y')S^{-1}(x''')x''y'' = \boldsymbol{\lambda}((x'y)')S^{-1}(x'')(x'y)'' = \boldsymbol{\lambda}(x'y)S^{-1}(x'')$.[5] It then follows that $\widehat{\boldsymbol{\phi}}(\widehat{\boldsymbol{\phi}}^{-1}(x)) = \widehat{\boldsymbol{\phi}}(\boldsymbol{\lambda}(S(x)?)) = \boldsymbol{\lambda}(S(x)\boldsymbol{\Lambda}')\boldsymbol{\Lambda}'' \overset{\text{by (2.3)}}{=} \boldsymbol{\lambda}(S(x)'\boldsymbol{\Lambda})S^{-1}(S(x)'') = \boldsymbol{\lambda}(\epsilon(S(x)')\boldsymbol{\Lambda})S^{-1}(S(x)'') = S^{-1}(\epsilon(S(x)')S(x)'') = x$. Similarly, we calculate $\widehat{\boldsymbol{\phi}}^{-1}(\widehat{\boldsymbol{\phi}}(\beta)) = \widehat{\boldsymbol{\phi}}^{-1}(\beta(\boldsymbol{\Lambda}')\boldsymbol{\Lambda}'')\beta(\boldsymbol{\Lambda}')\boldsymbol{\lambda}(S(\boldsymbol{\Lambda}'')?) = \beta(\boldsymbol{\lambda}(S(\boldsymbol{\Lambda}'')?)\boldsymbol{\Lambda}') = \beta(\boldsymbol{\lambda}(S(\boldsymbol{\Lambda})'?)S^{-1}(S(\boldsymbol{\Lambda})'')) \overset{\text{by (2.3)}}{=} \beta(\boldsymbol{\lambda}(S(\boldsymbol{\Lambda})?')?'') = \beta(\boldsymbol{\lambda}(\boldsymbol{\Lambda}?')?'') = \beta(\boldsymbol{\lambda}(\epsilon(?')\boldsymbol{\Lambda})?'') = \beta(\epsilon(?')?'') = \beta$.

We next show that $\widehat{\boldsymbol{\phi}}$ intertwines the left-U-module structures on U^* and U. With the left-U-module structure on U^* given by $x \rightharpoonup \beta = \beta(S(x)?)$, we must prove that $\beta(S(x)\boldsymbol{\Lambda}')\boldsymbol{\Lambda}'' = x\beta(\boldsymbol{\Lambda}')\boldsymbol{\Lambda}''$, or $\beta(x\boldsymbol{\Lambda}')\boldsymbol{\Lambda}'' = S^{-1}(x)\beta(\boldsymbol{\Lambda}')\boldsymbol{\Lambda}''$. But we have $\beta(x\boldsymbol{\Lambda}')\boldsymbol{\Lambda}'' = \beta(\epsilon(x')x''\boldsymbol{\Lambda}')\boldsymbol{\Lambda}'' = \beta((x'\boldsymbol{\Lambda})')S^{-1}(x'')(x'\boldsymbol{\Lambda})'' = \beta(\epsilon(x')\boldsymbol{\Lambda}')S^{-1}(x''')\epsilon(x'')\boldsymbol{\Lambda}'' = S^{-1}(x)\beta(\boldsymbol{\Lambda}')\boldsymbol{\Lambda}''$. ∎

2.3. Traces and the Radford map. For any irreducible representation $\mathcal{X}$ of a quantum group U, the ("quantum") trace in (A.10) is an invariant functional on U, i.e., an element of $\mathsf{Ch}(U)$ (see **A.1**).[6] (But the space of q-characters Ch is *not* spanned by q-traces over irreducible modules, as we have noted.) The Radford map sends each of the $\mathrm{Tr}_{\mathcal{X}}(\boldsymbol{g}^{-1}?)$ functionals into the center Z of U:

$$\widehat{\boldsymbol{\phi}} : \mathrm{Tr}_{\mathcal{X}}(\boldsymbol{g}^{-1}?) \to \widehat{\boldsymbol{\phi}}(\mathcal{X}) \in \mathsf{Z}. \tag{2.4}$$

[4] We use Sweedler's notation $\Delta(x) = \sum_{(x)} x'\,x''$ (see, e.g., [30]) with the summation symbols omitted in most cases; the defining property of the integral, for example, is then written as $\boldsymbol{\lambda}(x')x'' = \boldsymbol{\lambda}(x)$.

[5] Here and in what follows, we use the definitions of the antipode and counit written in the form (see, e.g., [30]) $x'S(x'') = S(x')x'' = \epsilon(x)1$ and $x'\epsilon(x'') = \epsilon(x')x'' = x$. Then, in particular, $x'S^{-1}(x''')x'' = x$.

[6] The reader not inclined to follow the details of the definition of $\boldsymbol{g}$ in (A.10) may think of it as just the element that makes the trace "quantum," i.e., invariant under the coadjoint action of the quantum group.

It suffices to have $\mathfrak{X}$ range the irreducible representations of U, because traces "see" only irreducible subquotients in indecomposable representations. As long as the linear span of q-traces over irreducible modules is not all of the space of q-characters, the Radford-map image of irreducible representations does not cover the center.

For any central element $a \in \mathsf{Z}$, its action on an irreducible representation $\mathfrak{X}$ is given by multiplication with a scalar, to be denoted by $a_{\mathfrak{X}} \in \mathbb{C}$. By the Radford map properties, we have the relation

$$a\,\widehat{\boldsymbol{\phi}}(\mathfrak{X}) = a_{\mathfrak{X}}\widehat{\boldsymbol{\phi}}(\mathfrak{X})$$

in the center. In particular, the Radford-map image of (traces over) all irreducible representations is the annihilator of the radical in the center.

2.3.1. For $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$, it is not difficult to verify that the right integral and the two-sided cointegral are given by

$$\boldsymbol{\lambda}(F^j E^m K^n) = \frac{1}{\zeta}\,\delta_{j,p-1}\delta_{m,p-1}\delta_{n,p+1}, \tag{2.5}$$

$$\boldsymbol{\Lambda} = \zeta\, F^{p-1}E^{p-1}\sum_{j=0}^{2p-1} K^j, \tag{2.6}$$

where we choose the normalization factor $\zeta = \sqrt{\frac{p}{2}}\,\frac{1}{([p-1]!)^2}$ [9].[7] For $\mathfrak{U}_{p,p'}$, the expressions for $\boldsymbol{\lambda}$ and $\boldsymbol{\Lambda}$ in [11] also hinge on the fact that $p-1$ is the highest nonzero power of the off-diagonal quantum group generators.

2.4. Comodulus. Another general notion that we need is that of a comodulus. For a right integral $\boldsymbol{\lambda}$, the *comodulus* "measures" how much $\boldsymbol{\lambda}$ differs from a left integral (see [32]): it is an element $\boldsymbol{a} \in U$ such that

$$(\mathrm{id}\otimes\boldsymbol{\lambda})\Delta(x) = \boldsymbol{\lambda}(x)\boldsymbol{a} \quad \forall x \in U.$$

A simple calculation then shows that the $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$ comodulus is $\boldsymbol{a} = K^2$. For $\mathfrak{U}_{p,p'}$, the comodulus is $\boldsymbol{a} = K^{2p-2p'}$.

3. Quantum group modules: from irreducible to projective

Irreducible (simple) and projective quantum group representations are considered below. By general philosophy of the Kazhdan–Lusztig duality, the irreducible quantum-group representations somehow "correspond" to irreducible chiral algebra representations in logarithmic conformal models. In particular, the Grothendieck ring is generally related to fusion in conformal field theory. While direct calculation of the fusion of chiral algebra

[7] In general, the (co)integral is defined up to a nonzero factor, but factorizable ribbon quantum groups offer a "canonical" normalization, derived from the condition $\mathcal{S}^2 = \mathrm{id}$ on the center; in accordance with (1.7), the normalization of $\mathcal{S}$ is inherited from the normalization of $\widehat{\boldsymbol{\phi}}$, and hence from that of the cointegral.

representations is typically quite difficult, this Grothendieck-ring structure may be considered a poor man's fusion (there is evidence that it is not totally meaningless). Apart from irreducible representations, their projective covers play an important role. To be specific, we now describe some aspects of the representation theory in the example of $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$.

3.1. Irreducible representations and the Grothendieck ring. There are $2p$ irreducible $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$-representations $\mathfrak{X}_r^{\pm}$, which can be conveniently labeled by the $\pm$ and $1 \leqslant r \leqslant p$. The highest-weight vector $|r\rangle^{\pm}$ of $\mathfrak{X}_r^{\pm}$ is annihilated by E, and its weight is determined by $K|r\rangle^{\pm} = \pm\mathfrak{q}^{r-1}|r\rangle^{\pm}$. The representation dimensions are $\dim \mathfrak{X}_r^{\pm} = r$. Some readers might find it suggestive to visualize the representations $\mathfrak{X}_r^{\pm}$ arranged into a "Kac table," a single row of boxes labeled by $r = 1, \dots, p$, each carrying a "$+$" and a "$-$" representation: $\underbrace{\square\square\square\dots\square\square}_{p}$.

We next recall that the Grothendieck ring is the free Abelian group generated by symbols $[M]$, where M ranges over all representations subject to relations $[M] = [M'] + [M'']$ for all exact sequences $0 \to M' \to M \to M'' \to 0$. Multiplication in the ring is induced by the tensor product of representations, with any indecomposable module occurring in the tensor product replaced by a sum of its simple subquotients.

$\underline{\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)}$. The Grothendieck ring of $\overline{\mathfrak{U}}_{\mathfrak{q}} s\ell(2)$ is (rather straightforwardly [9]) found to be given by

$$\mathfrak{X}_r^{\alpha}\mathfrak{X}_s^{\alpha'} = \sum_{\substack{t=|r-s|+1\\ \text{step}=2}}^{r+s-1} \widetilde{\mathfrak{X}}_t^{\alpha\alpha'} \tag{3.1}$$

where

$$\widetilde{\mathfrak{X}}_r^{\alpha} = \begin{cases} \mathfrak{X}_r^{\alpha}, & 1 \leqslant r \leqslant p,\\ \mathfrak{X}_{2p-r}^{\alpha} + 2\mathfrak{X}_{r-p}^{-\alpha}, & p+1 \leqslant r \leqslant 2p-1. \end{cases}$$

It can also be described in terms of Chebyshev polynomials, as the quotient of the polynomial ring $\mathbb{C}[x]$ over the ideal generated by the polynomial

$$\widehat{\Psi}_{2p}(x) = U_{2p+1}(x) - U_{2p-1}(x) - 2,$$

where $U_s(x)$ are Chebyshev polynomials of the second kind:

$$U_s(2\cos t) = \frac{\sin st}{\sin t}, \quad s \geqslant 1.$$

They satisfy the recurrence relations $xU_s(x) = U_{s-1}(x) + U_{s+1}(x)$, $s \geqslant 2$, with the initial data $U_1(x) = 1$, $U_2(x) = x$. Moreover, let

$$P_s(x) = \begin{cases} U_s(x), & 1 \leqslant s \leqslant p,\\ \frac{1}{2}U_s(x) - \frac{1}{2}U_{2p-s}(x), & p+1 \leqslant s \leqslant 2p. \end{cases} \tag{3.2}$$

Under the quotient map, the image of each P_s coincides with $\mathfrak{X}_s^{+}$ for $1 \leqslant s \leqslant p$ and with $\mathfrak{X}_{s-p}^{-}$ for $p+1 \leqslant s \leqslant 2p$.

The algebra in (3.1) is a nonsemisimple Verlinde algebra (commutative associative algebra with nonnegative integer structure coefficients, see [33]), with a unit given by $\mathcal{X}_1^+$. The algebra contains the ideal V_{p+1} generated by $\mathcal{X}_{p-r}^+ + \mathcal{X}_r^-$ with $1 \leqslant r \leqslant p-1$, $\mathcal{X}_p^+$, and $\mathcal{X}_p^-$. The quotient over V_{p+1} is a semisimple Verlinde algebra and in fact coincides with the fusion of the unitary $\widehat{s\ell}(2)$ representations of level $p-2$.[8]

The same algebra was derived in [19] from modular transformations of the triplet W-algebra characters in logarithmic $(p,1)$-models within a nonsemisimple generalization of the Verlinde formula (also see [34] for comparison with other derivations).

$\underline{\mathcal{U}_{p,p'}}$. The quantum group $\mathcal{U}_{p,p'}$ dual to the (p,p') logarithmic model has $2pp'$ irreducible representations $\mathcal{X}_{r,r'}^{\pm}$, $1 \leqslant r \leqslant p$, $1 \leqslant r' \leqslant p'$, with $\dim \mathcal{X}_{r,r'}^{\pm} = rr'$. They can be considered arranged into a "Kac table" $p'\left\{\underbrace{\begin{array}{|c|c|c|c|c|c|}\hline & & & \cdots & & \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline & & & \cdots & & \\ \hline & & & \cdots & & \\ \hline \end{array}}_{p}\right.$, with each box carrying a "$+$" and a "$-$" representation.

The Grothendieck ring structure is given by [11]

$$\mathcal{X}_{r,r'}^{\alpha}\mathcal{X}_{s,s'}^{\beta} = \sum_{\substack{u=|r-s|+1\\ \text{step}=2}}^{r+s-1} \sum_{\substack{u'=|r'-s'|+1\\ \text{step}=2}}^{r'+s'-1} \widetilde{\mathcal{X}}_{u,u'}^{\alpha\beta}, \tag{3.3}$$

where

$$\widetilde{\mathcal{X}}_{r,r'}^{\alpha} = \begin{cases} \mathcal{X}_{r,r'}^{\alpha}, & 1 \leqslant r \leqslant p,\ 1 \leqslant r' \leqslant p', \\ \mathcal{X}_{2p-r,r'}^{\alpha} + 2\mathcal{X}_{r-p,r'}^{-\alpha}, & p+1 \leqslant r \leqslant 2p-1,\ 1 \leqslant r' \leqslant p', \\ \mathcal{X}_{r,2p'-r'}^{\alpha} + 2\mathcal{X}_{r,r'-p'}^{-\alpha}, & 1 \leqslant r \leqslant p,\ p'+1 \leqslant r' \leqslant 2p'-1, \\ \mathcal{X}_{2p-r,2p'-r'}^{\alpha} + 2\mathcal{X}_{2p-r,r'-p'}^{-\alpha} & \\ \quad + 2\mathcal{X}_{r-p,2p'-r'}^{-\alpha} + 4\mathcal{X}_{r-p,r'-p'}^{\alpha}, & p+1 \leqslant r \leqslant 2p-1,\ p'+1 \leqslant r' \leqslant 2p'-1. \end{cases}$$

This algebra is a quotient of $\mathbb{C}[x,y]$ as described in [11]. The radical in this nonsemisimple Verlinde algebra (with a unit given by $\mathcal{X}_{1,1}^+$) is generated by the algebra action on $\mathcal{X}_{p,p'}^+$; the quotient over the radical coincides with the fusion of the (p,p') Virasoro minimal model.

The above algebra is a viable candidate for the ("K_0-type") fusion of W-algebra representations in the logarithmic (p,p')-models (see [13, 26]).

3.2. Indecomposable modules.

3.2.1. Irreducible quantum-group modules can be "glued" together to produce indecomposable representations. Already for $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$, its indecomposable representations (which

[8] It may be worth emphasizing that a Verlinde algebra structure involves not only an associative commutative structure but also a distinguished basis (the above quotient is that of Verlinde algebras). In particular, the reconstruction of the Verlinde algebra from its block decomposition as an associative algebra (the structure of primitive idempotents and elements in the radical in the algebra) requires extra information, cf. [19].

have been classified, rather directly, in [10] or can be easily deduced from a more general analysis in [35]) are rather numerous. Apart from the projective modules, to be considered separately in **3.3**, indecomposable representations are given by families of modules $\mathcal{W}^{\pm}_r(n)$, $\mathcal{M}^{\pm}_r(n)$, and $\mathcal{O}^{\pm}_r(n, z)$ that can be respectively represented as

$$\begin{tikzcd}
\mathcal{X}^{\pm}_r \arrow[dr, "x_1^{\pm}"] & & \mathcal{X}^{\pm}_r \arrow[dl, "x_2^{\pm}"'] \arrow[dr, "x_1^{\pm}"] & & \cdots \arrow[dl, "x_2^{\pm}"'] \arrow[dr, "x_1^{\pm}"] & & \mathcal{X}^{\pm}_r \arrow[dl, "x_2^{\pm}"'] \\
& \mathcal{X}^{\mp}_{p-r} & & \mathcal{X}^{\mp}_{p-r} & \cdots & \mathcal{X}^{\mp}_{p-r} &
\end{tikzcd}$$

(with $1 \leqslant r \leqslant p-1$, and integer $n \geqslant 2$ the number of $\mathcal{X}^{\pm}_r$ modules),

$$\begin{tikzcd}
& \mathcal{X}^{\pm}_r \arrow[dl, "x_2^{\pm}"'] \arrow[dr, "x_1^{\pm}"] & & \cdots \arrow[dl, "x_2^{\pm}"'] \arrow[dr, "x_1^{\pm}"] & & \mathcal{X}^{\pm}_r \arrow[dl, "x_2^{\pm}"'] \arrow[dr, "x_1^{\pm}"] & \\
\mathcal{X}^{\mp}_{p-r} & & \mathcal{X}^{\mp}_{p-r} & \cdots & \mathcal{X}^{\mp}_{p-r} & & \mathcal{X}^{\mp}_{p-r}
\end{tikzcd}$$

(with $1 \leqslant r \leqslant p-1$, and integer $n \geqslant 2$ the number of $\mathcal{X}^{\mp}_{p-r}$ modules), and

$$\begin{tikzcd}
& \mathcal{X}^{\pm}_r \arrow[dl, "x_2^{\pm}"'] \arrow[dr, "x_1^{\pm}"] & & \mathcal{X}^{\pm}_r \arrow[dl, "x_2^{\pm}"'] & \cdots & \mathcal{X}^{\pm}_r \arrow[dr, "x_1^{\pm}"] & \\
\mathcal{X}^{\mp}_{p-r} & & \mathcal{X}^{\mp}_{p-r} & & \cdots & & \mathcal{X}^{\mp}_{p-r} \\
& & & \mathcal{X}^{\pm}_r \arrow[ulll, bend left, "z_1 x_1^{\pm}"] \arrow[urrr, bend right, "z_2 x_2^{\pm}"'] & & &
\end{tikzcd}$$

(with $1 \leqslant r \leqslant p-1$, $z = z_1 : z_2 \in \mathbb{CP}^1$, and integer $n \geqslant 1$ the number of the $\mathcal{X}^{\pm}_r$ modules). The small x_i^+ and x_i^-, $i = 1, 2$, are basis elements chosen in the respective spaces $\mathbb{C}^2 = \mathrm{Ext}^1_{\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)}(\mathcal{X}^+_r, \mathcal{X}^-_{p-r})$ and $\mathbb{C}^2 = \mathrm{Ext}^1_{\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)}(\mathcal{X}^-_{p-r}, \mathcal{X}^+_r)$; they in fact generate the algebra $\mathrm{Ext}^{\bullet}_s$ (with the Yoneda product) with the relations

$$x_i^+ x_j^+ = x_i^- x_j^- = x_1^+ x_2^- + x_2^+ x_1^- = x_1^- x_2^+ + x_2^- x_1^+ = 0$$

(see [10] for the details).

Interestingly, a very similar picture (the "zigzag," although not the "$\mathcal{O}$" modules) also occurred in a different context [36, 37].

3.2.2. The representation category decomposes into subcategories as follows. For $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$, the familiar ("quantum") Casimir element

$$\boldsymbol{C} = (\mathfrak{q} - \mathfrak{q}^{-1})^2 EF + \mathfrak{q}^{-1} K + \mathfrak{q} K^{-1} \tag{3.4}$$

satisfies the minimal polynomial relation $\Psi_{2p}(\boldsymbol{C}) = 0$, where [9]

$$\Psi_{2p}(x) = (x - \beta_0)\,(x - \beta_p) \prod_{s=1}^{p-1} (x - \beta_s)^2, \quad \beta_s = \mathfrak{q}^s + \mathfrak{q}^{-s}. \tag{3.5}$$

This relation yields a decomposition of the representation category into the direct sum of full subcategories $\mathcal{C}(s)$ such that $(\boldsymbol{C} - \beta_s)$ acts nilpotently on objects in $\mathcal{C}(s)$. Because $\beta_s \neq \beta_{s'}$ for $0 \leqslant s \neq s' \leqslant p$, there are $p+1$ full subcategories $\mathcal{C}(s)$ for $0 \leqslant s \leqslant p$. Each $\mathcal{C}(s)$ with $1 \leqslant s \leqslant p-1$ contains precisely two irreducible modules $\mathcal{X}^+_s$ and $\mathcal{X}^-_{p-s}$ (because the Casimir element acts by multiplication with β_s on precisely these two) and infinitely many indecomposable modules. The irreducible modules $\mathcal{X}^+_p$ and $\mathcal{X}^-_p$ corresponding to the respective eigenvalues β_p and β_0 comprise the respective categories $\mathcal{C}(p)$ and $\mathcal{C}(0)$.

3.3. Projective modules. The process of constructing the extensions stops at projective modules — projective covers of each irreducible representation. Taking direct sums of projective modules then gives projective covers of all indecomposable representation.

A few irreducible representations are their own projective covers; these are $\mathcal{X}^\pm_p$ for $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ and $\mathcal{X}^\pm_{p,p'}$ for $\mathcal{U}_{p,p'}$. The other irreducible representations have projective covers filtered by several irreducible subquotients.

For $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$, the projective cover $\mathcal{P}^\pm_r$ of $\mathcal{X}^\pm_r$, $r = 1, \dots, p-1$, can be represented as[9]

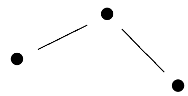
(3.6)
$$\begin{tikzcd}[column sep=small]
& \overset{\mathcal{X}^\pm_r}{\bullet} \arrow[dl] \arrow[dr] & \\
\overset{\mathcal{X}^\mp_{p-r}}{\bullet} \arrow[dr] & & \overset{\mathcal{X}^\mp_{p-r}}{\bullet} \arrow[dl] \\
& \overset{\mathcal{X}^\pm_r}{\bullet} &
\end{tikzcd}$$

It follows that $\dim \mathcal{P}^\pm_r = 2p$. For $\mathcal{U}_{p,p'}$, besides 2 irreducible projective modules of dimension pp', there are $2(p-1+p'-1)$ projective modules of dimension $2pp'$ and $2(p-1)(p'-1)$ projective modules of dimension $4pp'$ (see [11], where a diagram with 16 subquotients is also given).

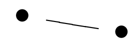
Regarding this picture for projective modules (as well as more involved pictures in [11]), it is useful to keep in mind that because of the periodicity in powers of $\mathfrak{q}$, the top and the bottom subquotients sit in the same grade (measured by eigenvalues of the Cartan generator K), as do the two "side" subquotients. A picture that makes this transparent and which shows the states in projective modules can be drawn as follows. Taking $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ with $p = 5$ and choosing $r = 3$ for example, we first represent the $\mathcal{X}^-_r = \mathcal{X}^-_3$ and $\mathcal{X}^+_{p-r} = \mathcal{X}^+_2$

[9] In diagrams of this type, first, the arrows are directed towards submodules; second, it is understood that the quantum group action on each irreducible representation is changed in agreement with the arrows connecting a given subquotient with others. This is of course true for the "two-floor" indecomposable modules considered above, but is even more significant for the projective modules, where the $\overset{\mathcal{X}^\pm_r}{\bullet} \longrightarrow \overset{\mathcal{X}^\mp_{p-r}}{\bullet}$ extensions alone do not suffice to describe the quantum group action. Constructing the quantum group action there requires some more work, but is not very difficult for each of the quantum groups considered here, as explicit formulas in [9, 11] show.

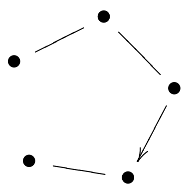
irreducible modules as



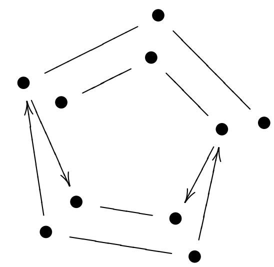
and



and then construct their extension



which actually gives a Verma module (the arrow is directed to the submodule). From this module and a contragredient one, we further construct the projective module $\mathcal{P}_2^+$ as the extension



where pairs of nearby dots represent states that actually sit in the same grade.

3.3.1. From the Grothendieck ring to the tensor algebra. The results in [35] go beyond the Grothendieck ring for the quantum group closely related to $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$: tensor products of the indecomposable representations are evaluated there. It follows from [35] that the $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ Grothendieck ring (3.1) is in fact the result of "forceful semisimplification" of the following tensor product algebra of irreducible representations. First, if $r + s - p \leqslant 1$, then, obviously, only irreducible representations occur in the decomposition:

$$\mathcal{X}_r^\alpha \otimes \mathcal{X}_s^\beta = \bigoplus_{\substack{t=|r-s|+1\\ \text{step}=2}}^{r+s-1} \mathcal{X}_t^{\alpha\beta}$$

(the sum contains $\min(r, s)$ terms). Next, if $r + s - p \geqslant 2$ and is even, $r + s - p = 2n$ with $n \geqslant 1$, then

$$\mathcal{X}_r^\alpha \otimes \mathcal{X}_s^\beta = \bigoplus_{\substack{t=|r-s|+1\\ \text{step}=2}}^{2p-r-s-1} \mathcal{X}_t^{\alpha\beta} \oplus \bigoplus_{a=1}^{n} \mathcal{P}_{p+1-2a}^{\alpha\beta}.$$

Finally, if $r + s - p \geqslant 3$ and is odd, $r + s - p = 2n + 1$ with $n \geqslant 1$, then

$$\mathcal{X}_r^\alpha \otimes \mathcal{X}_s^\beta = \bigoplus_{\substack{t=|r-s|+1\\ \text{step}=2}}^{2p-r-s-1} \mathcal{X}_t^{\alpha\beta} \oplus \bigoplus_{a=0}^{n} \mathcal{P}_{p-2a}^{\alpha\beta}.$$

We note that in each of the last two formulas, the first sum in the right-hand side contains $p - \max(r, s)$ terms, and therefore disappears whenever $\max(r, s) = p$ (see [35] for the tensor products of other modules in **3.2.1**).

3.3.2. Remarks.

(1) It follows that the irreducible $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ representations produce only (themselves and) projective modules in the tensor algebra. Because tensor products of any

modules with projective modules decompose into projective modules, we can consistently restrict ourself to only the irreducible and projective modules (in other words, there is a subring in the tensor algebra). This is a very special situation, however, specific to $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ (and the slightly larger algebra in [35]); generically, indecomposable representations other than the projective modules occur in tensor products of irreducible representations.[10] In particular, the true tensor algebra behind the Grothendieck ring in (3.3) is likely to involve various other indecomposable modules in the product of irreducible representations. Already for $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$, specifying the *full* tensor algebra means evaluating the products of all the representations listed in **3.2.1**.

(2) Reiterating the point in [10], we note that the previous remark fully applies to fusion of the chiral algebra (triplet W-algebra [38, 8, 27]) representations in $(p, 1)$ logarithmic conformal field theory models, once the fusion is taken not in the K_0-version but with "honest" indecomposable representations [24, 25]. While it is possible to consider such a fusion of only irreducible and projective W-algebra modules, the full fusion algebra must include all of the "$\mathcal{W}$," "$\mathcal{M}$," and "$\mathcal{O}$" indecomposable modules of the triplet algebra (with the last ones, somewhat intriguingly, being dependent on $z \in \mathbb{CP}^1$).

3.3.3. Pseudotraces. Projective modules serve another, somewhat technical but useful purpose. It was noted in **2.3** that traces over irreducible representations do not span the entire space Ch of q-characters. Projective modules provide what is missing: they allow constructing pseudotraces $\mathrm{Tr}_{\mathbb{P}}(\boldsymbol{g}^{-1}?\sigma)$ (for certain maps and modules $\sigma : \mathbb{P} \to \mathbb{P}$) that together with the traces $\mathrm{Tr}_{\mathcal{X}}(\boldsymbol{g}^{-1}?)$ over irreducible representations span all of Ch: a basis γ_A in Ch can be constructed such that with a subset of the γ_A is given by traces over irreducible representations and the rest by pseudotraces associated with projective modules in each full subcategory.

The strategy for constructing the pseudotraces is as follows. For any (reducible) module $\mathbb{P}$ and a map $\sigma : \mathbb{P} \to \mathbb{P}$, the functional

$$\gamma : x \mapsto \mathrm{Tr}_{\mathbb{P}}(\boldsymbol{g}^{-1} x \sigma) \tag{3.7}$$

is a q-character if and only if (cf. (A.2))

$$0 = \gamma(xy) - \gamma(S^2(y)x) \equiv \mathrm{Tr}_{\mathbb{P}}(\boldsymbol{g}^{-1} x [y, \sigma]). \tag{3.8}$$

It is possible to find reducible indecomposable modules $\mathbb{P}$ and maps σ satisfying (3.8). This requires taking $\mathbb{P}$ to be the projective module in a chosen full subcategory (one of those containing more than one module). For $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$, this is

$$\mathbb{P}_r = \mathcal{P}^+_r \oplus \mathcal{P}^-_{p-r}, \quad 1 \leqslant r \leqslant p-1, \tag{3.9}$$

[10] I thank V. Schomerus for this remark and a discussion of this point.

and for $\mathcal{U}_{p,p'}$ this is the direct sum

$$\mathbb{P}_{r,r'} = \mathcal{P}^{+}_{r,r'} \oplus \mathcal{P}^{-}_{p-r,r'} \oplus \mathcal{P}^{-}_{r,p'-r'} \oplus \mathcal{P}^{+}_{p-r,p'-r'}, \tag{3.10}$$

plus the "boundary" cases where either $r = p$ or $r' = p'$, with two terms in the sum (here, $\mathcal{P}^{\pm}_{r,r'}$ is the projective cover of the irreducible representation $\mathcal{X}^{\pm}_{r,r'}$). In all cases, σ is a linear map that sends the bottom module in the filtration of each projective module into "the same" module at a higher level in the filtration. Such maps are not defined uniquely (e.g., they depend on the choice of the bases and, obviously, on the "admixture" of lower-lying modules in the filtration), but anyway, taken together with the traces over irreducible representations, they allow constructing a basis in Ch. For $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$, there is a single pseudotrace for each r in (3.9) obtained by letting σ send the bottom of both diamonds of type (3.6) into the top. This gives just $p-1$ linearly independent elements of Ch. The structure for $\mathcal{U}_{p,p'}$ is somewhat richer, and the counting goes as follows [11]. There are not one but three other copies of the bottom subquotient in each of the four projective modules in (3.10). For the 12 parameters thus emerging, 7 constraints follow from (3.8). Of the remaining 5 different maps satisfying (3.8), there is just one (the map to the very top) for each of the $\frac{1}{2}(p-1)(p'-1)$ modules of form (3.10), two for each of the $\frac{1}{2}(p-1)p'$ modules, and two more for each of the $\frac{1}{2}p(p'-1)$ modules. This gives the total of

$$\frac{1}{2}(p-1)(p'-1) + (p-1)p' + p(p'-1)$$

linearly independent pseudotraces. Together with the traces over $2pp'$ irreducible representations, we thus obtain $\frac{1}{2}(3p-1)(3p'-1)$ linearly independent elements of Ch.

3.3.4. "Radford" basis. Radford-map images of the basis γ_A of traces and pseudotraces in Ch give a basis

$$\widehat{\boldsymbol{\phi}}_A = \widehat{\boldsymbol{\phi}}(\gamma_A)$$

in the quantum-group center Z. This basis plays an important role in what follows, being one of the two special bases related by $\mathcal{S} \in SL(2,\mathbb{Z})$. The other special basis is associated with the Drinfeld map considered in the next section.

3.3.5. Projective modules and the center. Projective modules are also a crucial ingredient in finding the quantum group center. Central elements are in a $1:1$ correspondence with *bimodule* endomorphism of the regular representation. We recall that viewed as a left module, the regular representation decomposes into projective modules, each entering with the multiplicity given by the dimension of its simple quotient. Generalizing this picture to a bimodule decomposition shows that the multiplicities are in fact tensor factors with respect to the right action. A typical block of the bimodule decomposition of the regular representation looks as follows: with respect to the left action, it is a sum of projective modules in one full subcategory, with each projective (externally) tensored

with a suitable simple module. For $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$, where the subquotients are few and therefore the picture is not too complicated, it can be drawn as [9]

$$\begin{tikzcd}
& \mathcal{X}_r^+ \boxtimes \mathcal{X}_r^+ \arrow[dl] \arrow[dr] & & & \mathcal{X}_{p-r}^- \boxtimes \mathcal{X}_{p-r}^- \arrow[dl] \arrow[dr] & \\
\mathcal{X}_{p-r}^- \boxtimes \mathcal{X}_r^+ \arrow[dr] & & \mathcal{X}_{p-r}^- \boxtimes \mathcal{X}_r^+ \arrow[dl] & \mathcal{X}_r^+ \boxtimes \mathcal{X}_{p-r}^- \arrow[dr] & & \mathcal{X}_r^+ \boxtimes \mathcal{X}_{p-r}^- \arrow[dl] \\
& \mathcal{X}_r^+ \boxtimes \mathcal{X}_r^+ & & & \mathcal{X}_{p-r}^- \boxtimes \mathcal{X}_{p-r}^- &
\end{tikzcd}$$

With respect to the right action, the picture is totally symmetric, but with the subquotients placed as above, the structure of their extensions has to be drawn as

$$\begin{tikzcd}
& \mathcal{X}_r^+ \boxtimes \mathcal{X}_r^+ \arrow[drrr] \arrow[drrrrr] & & & & \mathcal{X}_{p-r}^- \boxtimes \mathcal{X}_{p-r}^- \arrow[dlllll] \arrow[dlll] & \\
\mathcal{X}_{p-r}^- \boxtimes \mathcal{X}_r^+ \arrow[drrrrr] & & \mathcal{X}_{p-r}^- \boxtimes \mathcal{X}_r^+ \arrow[drrr] & & \mathcal{X}_r^+ \boxtimes \mathcal{X}_{p-r}^- \arrow[dlll] & & \mathcal{X}_r^+ \boxtimes \mathcal{X}_{p-r}^- \arrow[dlllll] \\
& \mathcal{X}_r^+ \boxtimes \mathcal{X}_r^+ & & & & \mathcal{X}_{p-r}^- \boxtimes \mathcal{X}_{p-r}^- &
\end{tikzcd}$$

Pictures of this type immediately yield the number of central elements *and their associative algebra structure*. First, each block yields a primitive idempotent $\boldsymbol{e}_I$, which is just the projector on this block; second, there are maps sending $A \boxtimes B$ bimodules into "the same" bimodules at lower levels, yielding nilpotent central elements. For $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$, the bimodule decomposition contains $p-1$ blocks of the above structure, plus two more given by $\mathcal{X}_p^+ \boxtimes \mathcal{X}_p^+$ and $\mathcal{X}_p^- \boxtimes \mathcal{X}_p^-$; in each of the "complicated" blocks, there are two bimodule automorphisms under which either the top $\mathcal{X}_r^+ \boxtimes \mathcal{X}_r^+$ or the top $\mathcal{X}_{p-r}^- \boxtimes \mathcal{X}_{p-r}^-$ goes into the corresponding bottom one, yielding two two central elements $\boldsymbol{w}_r^+$ and $\boldsymbol{w}_r^-$ with zero products among themselves. Therefore, the $(3p-1)$-dimensional center decomposes into a direct sum of associative algebras as

$$\mathsf{Z}_{\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)} = \mathfrak{I}_p^{(1)} \oplus \mathfrak{I}_0^{(1)} \oplus \bigoplus_{r=1}^{p-1} \mathfrak{B}_r^{(3)},$$

where the dimension of each algebra is shown as a superscript.

For $\mathcal{U}_{p,p'}$, there are several intermediate levels in the filtration of projective modules, and hence the nilpotent elements are more numerous and have a nontrivial multiplication table (see [11] for the details); the center is $\frac{1}{2}(3p-1)(3p'-1)$-dimensional and decomposes into a direct sum of associative algebras as

$$\mathsf{Z}_{\mathcal{U}_{p,p'}} = \mathfrak{I}_{p,p'}^{(1)} \oplus \mathfrak{I}_{0,p'}^{(1)} \oplus \bigoplus_{r=1}^{p-1} \mathfrak{B}_{r,p'}^{(3)} \oplus \bigoplus_{r'=1}^{p'-1} \mathfrak{B}_{p,r'}^{(3)} \oplus \bigoplus_{r,r' \in \mathcal{I}_1} \mathfrak{A}_{r,r'}^{(9)}, \tag{3.11}$$

where the dimension of each algebra is shown as a superscript (and where $|\mathcal{I}_1| = \frac{1}{2}(p-1)(p'-1)$). The $\mathfrak{B}^{(3)}$ algebras are just as in the previous formula, and each $\mathfrak{A}_{r,r'}^{(9)}$ is spanned by a primitive

idempotent $\boldsymbol{e}_{r,r'}$ (acting as identity on $\mathfrak{A}^{(9)}_{r,r'}$) and eight radical elements $\boldsymbol{v}^{\nearrow}_{r,r'}$, $\boldsymbol{v}^{\swarrow}_{r,r'}$, $\boldsymbol{v}^{\nwarrow}_{r,r'}$, $\boldsymbol{v}^{\searrow}_{r,r'}$, $\boldsymbol{w}^{\uparrow}_{r,r'}$, $\boldsymbol{w}^{\rightarrow}_{r,r'}$, $\boldsymbol{w}^{\downarrow}_{r,r'}$, $\boldsymbol{w}^{\leftarrow}_{r,r'}$ that have the nonzero products

$$\boldsymbol{v}^{\nearrow}_{r,r'}\boldsymbol{v}^{\nwarrow}_{r,r'} = \boldsymbol{w}^{\uparrow}_{r,r'}, \qquad \boldsymbol{v}^{\nearrow}_{r,r'}\boldsymbol{v}^{\searrow}_{r,r'} = \boldsymbol{w}^{\rightarrow}_{r,r'},$$
$$\boldsymbol{v}^{\swarrow}_{r,r'}\boldsymbol{v}^{\nwarrow}_{r,r'} = \boldsymbol{w}^{\leftarrow}_{r,r'}, \qquad \boldsymbol{v}^{\swarrow}_{r,r'}\boldsymbol{v}^{\searrow}_{r,r'} = \boldsymbol{w}^{\downarrow}_{r,r'}.$$

4. Drinfeld map and factorizable and ribbon structures

4.1. M-matrix and the Drinfeld map. For a quasitriangular Hopf algebra U with the universal R-matrix R, the M-matrix is the "square" of the R-matrix, defined as

$$M = R_{21}R_{12} \in U \otimes U.$$

It satisfies the relations

$$(4.1)\qquad \begin{aligned}(\Delta\otimes\mathrm{id})(M) &= R_{32}M_{13}R_{23},\\ M\Delta(x) &= \Delta(x)M \quad \forall x\in U.\end{aligned}$$

Indeed, using (A.5), we find $(\Delta\otimes\mathrm{id})(R_{21}) = R_{32}R_{31}$ and then using (A.4) we obtain the first relation in (4.1). Next, it follows from (A.3) that $R_{21}R_{12}\Delta(x) = (R_{12}\Delta(x))^{\mathrm{op}}R_{12} = (\Delta^{\mathrm{op}}(x)R_{12})^{\mathrm{op}}R_{12} = \Delta(x)R_{21}R_{12}$, that is, the second relation in (4.1).

The Drinfeld map $\boldsymbol{\chi} : U^* \to U$ is defined as

$$\boldsymbol{\chi} : \beta \mapsto (\beta\otimes\mathrm{id})(M),$$

that is, if we write the M-matrix as

$$(4.2)\qquad M = \sum_I \boldsymbol{m}_I \otimes \boldsymbol{n}_I,$$

then $\boldsymbol{\chi}(\beta) = \sum_I \beta(\boldsymbol{m}_I)\boldsymbol{n}_I$.

Whenever $\boldsymbol{\chi} : U^* \to U$ is an isomorphism of vector spaces, the Hopf algebra U is called *factorizable* [39]. Equivalently, this means that $\boldsymbol{m}_I$ and $\boldsymbol{n}_I$ in (4.2) are two *bases* in U.

4.1.1. Lemma ([40]). *In a factorizable Hopf algebra U, by restriction to* Ch (*see* **A.1**), *the Drinfeld map defines a homomorphism*

$$\mathsf{Ch}(U) \to \mathsf{Z}(U)$$

of associative algebras.

Proof. We first show that $\boldsymbol{\chi}(\beta)$ is central for any $\beta\in\mathsf{Ch}$: for any $x\in U$, we calculate $\boldsymbol{\chi}(\beta)x = \sum_I \beta(\boldsymbol{m}_I)\boldsymbol{n}_I x = \sum_I \beta(\boldsymbol{m}_I x'' S^{-1}(x'))\boldsymbol{n}_I x'''$. But because $M\Delta(x) = \Delta(x)M$ and $\beta(xy) = \beta(S^2(y)x)$, we obtain that $\boldsymbol{\chi}(\beta)x = \sum_I \beta(S(x')x''\boldsymbol{m}_I)x'''\boldsymbol{n}_I = x\boldsymbol{\chi}(\beta)$.

Next, to show that $\boldsymbol{\chi} : \mathsf{Ch} \to \mathsf{Z}$ is a homomorphism of associative algebras, we recall that the product of two functionals is defined as $\beta\gamma(x) = (\beta\otimes\gamma)(\Delta(x))$, and therefore, using the first

relation in (4.1), we have $\boldsymbol{\chi}(\beta\gamma) = (\beta\otimes\gamma\otimes\mathrm{id})((\Delta\otimes\mathrm{id})(M)) = (\beta\otimes\gamma\otimes\mathrm{id})(R_{32}M_{13}R_{23}) = (\gamma\otimes\mathrm{id})(R_{21}\boldsymbol{\chi}(\beta)R_{12}) = \boldsymbol{\chi}(\beta)(\gamma\otimes\mathrm{id})(R_{21}R_{12}) = \boldsymbol{\chi}(\beta)(\gamma\otimes\mathrm{id})(M) = \boldsymbol{\chi}(\beta)\boldsymbol{\chi}(\gamma)$.[11] □

For the quantum groups considered here, the above homomorphism is in fact an *isomorphism* (cf. [40, 41]).

4.2. Kazhdan–Lusztig-dual quantum groups: Drinfeld's double, M-matrix, and R-matrix. The quantum groups U originating from logarithmic conformal models *are not quasitriangular*, but are nevertheless factorizable in the following sense: the M-matrix can be expressed through an R that is the universal R-matrix of a somewhat larger quantum group $\bar{D}$.[12] This is true for both $\overline{\mathfrak{U}}_{\mathfrak{q}}s\ell(2)$ and $\mathfrak{U}_{p,p'}$, with the extension to $\bar{D}$ realized in each case by introducing the generator $k = K^{1/2}$. In other words, in each case, there is a *quasitriangular* quantum group $\bar{D}$ with a set of generators $k,\dots$, with a universal R-matrix R, such that $R_{21}R_{12}$ turns out to belong to $U\otimes U$, where U is the Hopf subalgebra in $\bar{D}$ generated by $K = k^2$ and the other $\bar{D}$ generators. In the respective cases, U is either $\overline{\mathfrak{U}}_{\mathfrak{q}}s\ell(2)$ or $\mathfrak{U}_{p,p'}$.

The universal R-matrix for $\bar{D}$, in turn, comes from constructing the Drinfeld double [42] of the quantum group B generated by screenings in the logarithmic conformal field model [9, 13].[13] For $(p,1)$ models, B is the Taft Hopf algebra with generators E and k, with $kEk^{-1} = \mathfrak{q}E$, $E^p = 0$, and $k^{4p} = 1$. We then take the dual space B^*, which is a Hopf algebra with the multiplication, comultiplication, unit, counit, and antipode given by

$$\begin{gathered}\langle\beta\gamma, x\rangle = \langle\beta, x'\rangle\langle\gamma, x''\rangle, \quad \langle\Delta(\beta), x\otimes y\rangle = \langle\beta, yx\rangle,\\ \langle 1, x\rangle = \epsilon(x), \quad \epsilon(\beta) = \langle\beta, 1\rangle, \quad \langle S(\beta), x\rangle = \langle\beta, S^{-1}(x)\rangle\end{gathered} \tag{4.3}$$

for $\beta,\gamma\in B^*$ and $x,y\in B$. The Drinfeld double $D(B)$ is a Hopf algebra with the underlying vector space $B^*\otimes B$ and with the multiplication, comultiplication, unit, counit, and antipode given by those in B, by Eqs. (4.3), and by

$$x\beta = \beta(S^{-1}(x''')?x')x'', \qquad x\in B, \quad \beta\in B^*. \tag{4.4}$$

The resulting Hopf algebra $D(B)$ is canonically endowed with the universal R-matrix [42].

The doubling procedure also introduces the dual $\varkappa$ to the Cartan element k, which is then to be eliminated by passing to the quotient over (the Hopf ideal generated by) $k\kappa - 1$

[11]It was noted in [41] that "Drinfeld's proof of [40, 3.3] shows more than what is actually stated in [40, 3.3]." It actually follows that $\boldsymbol{\chi}(\beta\gamma) = \boldsymbol{\chi}(\beta)\boldsymbol{\chi}(\gamma)$ whenever $\beta\in\mathsf{Ch}(U)$ and $\gamma\in U^*$.

[12]The standard definition of a factorizable quantum group [39] involves the universal R-matrix as well, which is the reason why we express some caution; the R-matrix in the M-matrix property (4.1) is *not* an element of $U\otimes U$. In particular, U is not unimodular in our case (but U^* is!).

[13]The screenings generate only the upper-triangular subalgebra of the Kazhdan–Lusztig-dual quantum group; to these upper-triangular subalgebra, we add Cartan generator(s) constructed from zero modes of the free fields involved in the chosen free-field realization. This gives the B quantum group.

(it follows that $k\kappa$ is central in the double). The quotient $\bar{D}$ is still quasitriangular, but evaluating the M-matrix and the ribbon element for it shows that they are turn out to be those for the (Hopf) subalgebra generated by $K \equiv k^2$ and the other $\bar{D}$ generators, which is finally the Kazhdan–Lusztig-dual quantum group. This was how the Kazhdan–Lusztig-dual quantum groups, together with the crucial structures on them, were derived in [9, 13].

For $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$, for example, the M-matrix is explicitly expressed in terms of the PBW basis as

$$M = \frac{1}{2p}\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\sum_{i=0}^{2p-1}\sum_{j=0}^{2p-1}\frac{(\mathfrak{q}-\mathfrak{q}^{-1})^{m+n}}{[m]![n]!}\mathfrak{q}^{m(m-1)/2+n(n-1)/2} \times \mathfrak{q}^{-m^2-mj+2nj-2ni-ij+mi}F^mE^nK^j\otimes E^mF^nK^i. \tag{4.5}$$

4.3. Drinfeld-map images of traces and pseudotraces. In a factorizable Hopf algebra, it follows that the Drinfeld-map images of the traces over irreducible representations form an algebra isomorphic to the Grothendieck ring. Thus, there are central elements

$$\boldsymbol{\chi}^{\pm}_r = \boldsymbol{\chi}(\mathrm{Tr}_{\mathcal{X}^{\pm}_r}(\boldsymbol{g}^{-1}?)), \qquad 1\leqslant r\leqslant p$$

for $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ and

$$\boldsymbol{\chi}^{\pm}_{r,r'} = \boldsymbol{\chi}(\mathrm{Tr}_{\mathcal{X}^{\pm}_{r,r'}}(\boldsymbol{g}^{-1}?)), \quad 1\leqslant r\leqslant p,\ 1\leqslant r'\leqslant p'$$

for $\mathcal{U}_{p,p'}$, which satisfy the respective algebra (3.1) and (3.3).

4.3.1. In $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$, for example, Eq. (4.5) allows us to calculate the $\boldsymbol{\chi}^{\alpha}_s$ explicitly,

$$\boldsymbol{\chi}^{\alpha}_s = \alpha^{p+1}(-1)^{s+1}\sum_{n=0}^{s-1}\sum_{m=0}^{n}(\mathfrak{q}-\mathfrak{q}^{-1})^{2m}\mathfrak{q}^{-(m+1)(m+s-1-2n)} \times \times \begin{bmatrix}s-n+m-1\\ m\end{bmatrix}\begin{bmatrix}n\\ m\end{bmatrix}E^mF^mK^{s-1+\beta p-2n+m} \tag{4.6}$$

(where $\beta = 0$ if $\alpha = +1$ and $\beta = 1$ if $\alpha = -1$). In particular, $\boldsymbol{\chi}^{+}_2 = -\boldsymbol{C}$, where $\boldsymbol{C}$ is the Casimir element, Eq. (3.4). The fact that the $\boldsymbol{\chi}^{\pm}_r$ given by (4.6) satisfy Grothendieck-ring relations (3.1) implies a certain q-binomial identity, see [9].

4.3.2. Remark. The $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ Casimir element satisfies the minimal polynomial relation $\Psi_{2p}(\boldsymbol{C}) = 0$, with Ψ_{2p} in (3.5). This relation, with $p-1$ multiplicity-2 roots of Ψ_{2p}, allows constructing a basis in the center Z of $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ consisting of primitive idempotents $\boldsymbol{e}_r$ and elements $\boldsymbol{w}_r$ in the radical of the associative commutative algebra Z [9] (see [23] and also [43, Ch. V.2]). For this, we define the polynomials

$$\psi_0(x) = (x-\beta_p)\prod_{r=1}^{p-1}(x-\beta_r)^2, \qquad \psi_p(x) = (x-\beta_0)\prod_{r=1}^{p-1}(x-\beta_r)^2,$$

$$\psi_s(x) = (x-\beta_0)\,(x-\beta_p)\prod_{\substack{r=1\\ r\neq s}}^{p-1}(x-\beta_r)^2, \quad 1\leqslant s\leqslant p-1,$$

where we recall that all β_j are distinct. Then the canonical elements in the radical of Z are

$$\boldsymbol{w}_s^{\pm} = \pi_s^{\pm}\boldsymbol{w}_s, \qquad 1\leqslant s\leqslant p-1,$$

where

$$\boldsymbol{w}_s = \frac{1}{\psi_s(\beta_s)}\big(\boldsymbol{C}-\beta_s\big)\psi_s(\boldsymbol{C})$$

and we introduce the projectors

$$\pi_s^{+} = \frac{1}{2p}\sum_{n=0}^{s-1}\sum_{j=0}^{2p-1}\mathfrak{q}^{(2n-s+1)j}K^j, \qquad \pi_s^{-} = \frac{1}{2p}\sum_{n=s}^{p-1}\sum_{j=0}^{2p-1}\mathfrak{q}^{(2n-s+1)j}K^j,$$

and the canonical central idempotents are given by

$$\boldsymbol{e}_s = \frac{1}{\psi_s(\beta_s)}\big(\psi_s(\boldsymbol{C}) - \psi_s'(\beta_s)\boldsymbol{w}_s\big), \quad 0\leqslant s\leqslant p,$$

where we formally set $\boldsymbol{w}_0 = \boldsymbol{w}_p = 0$. A similar construction exists for the center of $\mathcal{U}_{p,p'}$ [11], where, in particular, there are not two but four types of projectors $\boldsymbol{\pi}^{\uparrow}_{r,r'}$, $\boldsymbol{\pi}^{\leftarrow}_{r,r'}$, $\boldsymbol{\pi}^{\rightarrow}_{r,r'}$, and $\boldsymbol{\pi}^{\downarrow}_{r,r'}$; for either algebra, these are projectors on the weights occurring in irreducible modules in the full subcategory labeled by the subscript.

4.3.3. "Drinfeld" basis. Applied to the basis γ_A of traces and pseudotraces in Ch, the Drinfeld map gives a basis

$$\boldsymbol{\chi}_A = \boldsymbol{\chi}(\gamma_A)$$

in the center Z.

This "Drinfeld" basis (which is not defined uniquely because pseudotraces are not defined uniquely) specifies an explicit splitting of the associative commutative algebra Z into the Grothendieck ring and its linear complement. The products of the Grothendieck ring elements with elements from the complement may also be of significance in the Kazhdan–Lusztig context. The full algebra of q-characters (traces and pseudotraces) for $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$, mapped into the center by the Drinfeld map, is evaluated in [44]; it can be understood as a generalized fusion, to be compared with a recent calculation to this effect in the logarithmic $(p,1)$ models in [34].

Under $\mathcal{S}\in SL(2,\mathbb{Z})$ acting as in (1.7), clearly, the Drinfeld basis elements are mapped into the Radford basis,

$$\mathcal{S}:\boldsymbol{\chi}_A\mapsto\widehat{\boldsymbol{\phi}}_A.$$

Realizing $\mathcal{T}\in SL(2,\mathbb{Z})$ on the center requires yet another structure, the ribbon element.

4.4. Ribbon structure. A *ribbon element* [45] is a $\boldsymbol{v} \in \mathsf{Z}$ such that

$$\Delta(\boldsymbol{v}) = M^{-1}(\boldsymbol{v} \otimes \boldsymbol{v}),$$

with $\epsilon(\boldsymbol{v}) = 1$ and $S(\boldsymbol{v}) = \boldsymbol{v}$ (and $\boldsymbol{v}^2 = \boldsymbol{u}S(\boldsymbol{u})$, see **A.3**). The procedure for finding the ribbon element involves two steps: we first find the canonical element (A.7) (which involves the universal R-matrix for the larger, quasitriangular quantum group $\bar{D}$ mentioned in **4.2**) and then evaluate the balancing element $\boldsymbol{g}$ (see **A.4**) in accordance with Drinfeld's Lemma (A.11), from the comodulus obtained from the explicit expression for the integral (this is the job done by the comodulus). Then

$$\boldsymbol{v} = \boldsymbol{u}\boldsymbol{g}^{-1}. \tag{4.7}$$

It follows, again, that $\boldsymbol{v}$ is an element of a Hopf subalgebra in $\bar{D}$, which is $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$ or $\mathcal{U}_{p,p'}$.

4.4.1. For $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$, where $\boldsymbol{g} = K^{p+1}$, we have [9]

$$\boldsymbol{v} = \sum_{s=0}^{p} (-1)^{s+1} \mathfrak{q}^{-\frac{1}{2}(s^2-1)} \boldsymbol{e}_s + \sum_{s=1}^{p-1} (-1)^p \mathfrak{q}^{-\frac{1}{2}(s^2-1)} [s] \frac{\mathfrak{q} - \mathfrak{q}^{-1}}{\sqrt{2p}} \widehat{\boldsymbol{\varphi}}_s, \tag{4.8}$$

where $\boldsymbol{e}_s$ are the canonical idempotents in the center and

$$\widehat{\boldsymbol{\varphi}}_s = \frac{p-s}{p} \widehat{\boldsymbol{\phi}}{}^{+}_s - \frac{s}{p} \widehat{\boldsymbol{\phi}}{}^{-}_{p-s}, \quad 1 \leqslant s \leqslant p-1, \tag{4.9}$$

are nilpotent central elements expressed through the Radford-map images $\widehat{\boldsymbol{\phi}}{}^{\pm}_s$ of the (traces over) irreducible representations $\mathcal{X}^{\pm}_s$.

4.4.2. Remark. The above form of $\boldsymbol{v}$ implies that [10]

$$\boldsymbol{v} = e^{2i\pi L_0}$$

(where L_0 is the zero-mode Virasoro generator in the $(p, 1)$ logarithmic conformal model); in particular, the exponents involving s^2 in (4.8) are simply related to conformal dimensions of primary fields. Rather interestingly, the nonsemisimple action of L_0 on the lattice vertex operator algebra underlying the construction of the logarithmic $(p, 1)$ model is thus correlated with the decomposition of the ribbon element with respect to the central idempotents and nilpotents.

4.4.3. For $\mathcal{U}_{p,p'}$, the ribbon element is given by

$$\boldsymbol{v} = \sum_{(r,r')\in\mathcal{I}} e^{2i\pi\Delta_{r,r'}} \boldsymbol{e}_{r,r'} + \text{nilpotent terms},$$

where $\boldsymbol{e}_{r,r'}$ are the $\frac{1}{2}(p+1)(p'+1)$ primitive idempotents in the associative commutative algebra Z (the explicit form of the nilpotent terms being not very illuminating at this level of detail, see [11] for the full formula), and

$$\Delta_{r,r'} = \frac{(pr' - p'r)^2 - (p - p')^2}{4pp'}$$

are conformal dimensions of primary fields borrowed from the logarithmic model [13].

5. MODULAR GROUP ACTION

5.1. Defining the action. In defining the modular group action on the center we follow [21, 22, 23] with an insignificant variation in the definition of $\mathcal{T}$, introduced in [9, 13] in order to simplify comparison with the modular group representation generated from characters of the chiral algebra in the corresponding logarithmic conformal model. On the quantum group center, the $SL(2,\mathbb{Z})$-action is defined by

$$\begin{aligned}&\mathcal{S}: x \mapsto \widehat{\boldsymbol{\phi}}\big(\boldsymbol{\chi}^{-1}(x)\big),\\&\mathcal{T}: x \mapsto e^{-i\pi\frac{c}{12}}\,\mathcal{S}\big(\boldsymbol{v}\mathcal{S}^{-1}(x)\big),\end{aligned} \tag{5.1}$$

where c is the central charge of the conformal model, e.g.,

$$c = 13 - 6\frac{p}{p'} - 6\frac{p'}{p}$$

for the (p,p') model.[14]

5.2. Calculation results. The result of evaluating (5.1) in each case gives the structure of the $SL(2,\mathbb{Z})$ representation of the type that was first noted in [23] for the small quantum $s\ell(2)$.[15]

5.2.1. On the center of $\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)$, the $SL(2,\mathbb{Z})$ representation is given by [9]

$$\mathsf{Z}_{\overline{\mathcal{U}}_{\mathfrak{q}} s\ell(2)} = R_{p+1} \oplus \mathbb{C}^2 \otimes R_{p-1}, \tag{5.2}$$

where $\mathbb{C}^2$ is the defining two-dimensional representation, $\mathcal{R}_{p-1}$ is a $(p-1)$-dimensional $SL(2,\mathbb{Z})$-representation (the "$\sin\frac{\pi rs}{p}$" representation, in fact, the one on the unitary $\widehat{s\ell}(2)_k$-characters at the level $k=p-2$), and $\mathcal{R}_{p+1}$ is a "$\cos\frac{\pi rs}{p}$" $(p+1)$-dimensional representation.

On the center of $\mathcal{U}_{p,p'}$, the $SL(2,\mathbb{Z})$-representation structure is given by [11]

$$\mathsf{Z}_{\mathcal{U}_{p,p'}} = R_{\min} \oplus R_{\mathrm{proj}} \oplus \mathbb{C}^2 \otimes (R_{\boxbslash} \oplus R_{\boxslash}) \oplus \mathbb{C}^3 \otimes R_{\min}, \tag{5.3}$$

where $\mathbb{C}^3$ is the symmetrized square of $\mathbb{C}^2$, $R_{\min}$ is the $\frac{1}{2}(p-1)(p'-1)$-dimensional $SL(2,\mathbb{Z})$-representation on the characters of the *rational* (p,p') Virasoro model, and R_{proj}, $R_{\boxbslash}$, and $R_{\boxslash}$ are certain $SL(2,\mathbb{Z})$ representations of the respective dimensions $\frac{1}{2}(p+1)(p'+1)$, $\frac{1}{2}(p+1)(p'-1)$, and $\frac{1}{2}(p-1)(p'+1)$.

As noted above, (5.2) and (5.3) coincide with the respective $SL(2,\mathbb{Z})$-representations on generalized characters of $(p,1)$ and (p,p') logarithmic conformal field models evaluated in [9, 13].

[14] Reversing the argument, for a factorizable ribbon quantum group that can be expected to correspond to a conformal field model, the normalization of $\mathcal{T}$ (i.e., the factor accompanying the ribbon element) may thus indicate the central charge, and the decomposition of the ribbon element into the basis of primitive idempotents and elements in the radical is suggestive about the conformal dimensions.

[15] The small quantum groups have been the subject of some constant interest, see, e.g., [46, 47, 48] and the references therein.

5.2.2. The role of the subrepresentations identified in (5.2) and (5.3) is yet to be understood from the quantum-group standpoint, but it is truly remarkable in the context of the Kazhdan–Lusztig correspondence. The occurrence of the $\mathbb{C}^n$ tensor factors is rigorously correlated with the fact that the $\psi_{b'}(\tau)$ functions in (1.1)–(1.2) are given by (certain linear combinations of) characters *times polynomials in* τ *of degree* $n-1$.

In the quantum group center, the subrepresentations in (5.2) and (5.3) are described as the span of certain combinations of the elements of "Radford" and "Drinfeld" bases $\widehat{\boldsymbol{\phi}}_A$ and $\boldsymbol{\chi}_A$. For $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$, in particular, the central elements (4.9), together with their $\mathcal{S}$-images $\frac{p-s}{p}\,\boldsymbol{\chi}_s^+ - \frac{s}{p}\,\boldsymbol{\chi}_{p-s}^-$, $1\leqslant s\leqslant p-1$, span the $\mathbb{C}^2\otimes R_{p-1}$ representation; in the logarithmic $(p,1)$ model, the same representation is realized on the $2(p-1)$ functions

$$\tau\Bigl(\frac{p-s}{p}\,\chi_s^+(\tau) - \frac{s}{p}\,\chi_{p-s}^-(\tau)\Bigr),\qquad \frac{p-s}{p}\,\chi_s^+(\tau) - \frac{s}{p}\,\chi_{p-s}^-(\tau),$$

where $\chi_r^{\pm}(\tau)$ are the triplet algebra characters [9]. On the other hand, the $(p+1)$-dimensional representation R_{p+1} in the center is linearly spanned by $\boldsymbol{\chi}_p^{\pm}$ and $\boldsymbol{\chi}_s^+ + \boldsymbol{\chi}_{p-s}^-$, $1\leqslant s\leqslant p-1$ (the ideal already mentioned after (3.2)); in the $(p,1)$ model, the same representation is realized on the linear combinations of characters

$$\chi_p^{\pm}(\tau),\qquad \chi_s^+(\tau)+\chi_{p-s}^-(\tau).$$

The $\mathcal{U}_{p,p'}$ setting in [11, Sec. **5.3**] gives rather an abundant picture of how the various traces *and pseudotraces*, mapped into the center, combine to produce the subrepresentations and how precisely these linear combinations correspond to the characters *and extended characters* in the logarithmic (p,p') model.[16] Here, we only note the R_{proj} representation, of dimension $\frac{1}{2}(p+1)(p'+1)$, linearly spanned by $\boldsymbol{\chi}_{r,r'}^+ + \boldsymbol{\chi}_{p-r,r'}^- + \boldsymbol{\chi}_{r,p'-r'}^- + \boldsymbol{\chi}_{p-r,p'-r'}^+$ (with $(r,r')\in\mathcal{I}_1$, where $|\mathcal{I}_1| = \frac{1}{2}(p-1)(p'-1)$), $\boldsymbol{\chi}_{r,p'}^+ + \boldsymbol{\chi}_{p-r,p'}^-$ (with $1\leqslant r\leqslant p-1$), $\boldsymbol{\chi}_{p,r'}^+ + \boldsymbol{\chi}_{p,p'-r'}^-$ ($1\leqslant r'\leqslant p'-1$), and $\boldsymbol{\chi}_{p,p'}^{\pm}$. In the logarithmic (p,p') model, the same $SL(2,\mathbb{Z})$-representation is realized on the linear combinations of W-algebra characters

$$\begin{aligned}
&\chi_{r,r'}(\tau)+2\chi_{r,r'}^+(\tau)+2\chi_{r,p'-r'}^-(\tau)+2\chi_{p-r,r'}^-(\tau)+2\chi_{p-r,p'-r'}^+(\tau), && (r,r')\in\mathcal{I}_1,\\
&2\chi_{p,p'-r'}^+(\tau)+2\chi_{p,r'}^-(\tau), && 1\leqslant r'\leqslant p'-1,\\
&2\chi_{p-r,p'}^+(\tau)+2\chi_{r,p'}^-(\tau), && 1\leqslant r\leqslant p-1,\\
&2\chi_{p,p'}^{\pm}(\tau)
\end{aligned}$$

(with the same size $|\mathcal{I}_1| = \frac{1}{2}(p-1)(p'-1)$ of the index set), where $\chi_{r,r'}(\tau)$ are the characters of the Virasoro *rational* model and $\chi_{r,r'}^{\pm}(\tau)$ are the other $2pp'$ characters of the W-algebra [13]. The above combinations do not involve generalized characters (which occur where the $\mathbb{C}^n$ factors are involved in the $SL(2,\mathbb{Z})$-representation isomorphic to the one in (5.3) and which are in fact the origin of these $\mathbb{C}^n$ factors from the conformal field theory standpoint).

[16] Once again: $\mathbb{C}$-linear combinations of the $\frac{1}{2}(3p-1)(3p'-1)$ traces and pseudotraces (mapped to the center) carry the same $SL(2,\mathbb{Z})$-representations as certain $\mathbb{C}[\tau]$-linear combinations of the $2pp' + \frac{1}{2}(p-1)(p'-1)$ *characters* of the W-algebra; the total dimension is $\frac{1}{2}(3p-1)(3p'-1)$ in either case.

A remarkable feature of the $SL(2,\mathbb{Z})$ representation on the $\mathcal{U}_{p,p'}$ center is the occurrence of $R_{\min}$, the $SL(2,\mathbb{Z})$-representation on the characters of the rational Virasoro model, even though the $\mathcal{U}_{p,p'}$-representations $\mathcal{X}^{\pm}_{r,r'}$ are in a $1:1$ correspondence not with all the primary fields of the W-algebra in the logarithmic model but just with those *except* the rational-model ones.

5.3. Beyond the quantum group. Two algebraic structures on the quantum group center are most important from the standpoint of the Kazhdan–Lusztig correspondence: the modular group action and the Grothendieck ring (the latter is a subring in the center spanned by Drinfeld-map images of the irreducible representations). The resulting Grothendieck rings, or Verlinde algebras are nonsemisimple.

A classification of Verlinde algebras has been proposed in a totally different approach, that of double affine Hecke algebras (Cherednik algebras) [49], where Verlinde algebras occur as certain representations of Cherednik algebras; an important point is that a modular group action is built into the structure of Cherednik algebras. It can thus be expected that the $(p,1)$-model fusion (the $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ Grothendieck ring) (3.1), of dimension $2p$, admits a realization associated with a Cherednik algebra representation. But because an isomorphic image of the Grothendieck ring is contained in the center, a natural further question is whether the entire $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$ center, of dimension $3p-1$, endowed with the $SL(2,\mathbb{Z})$ action, is also related to Cherednik algebras.

It was shown in [50] that the center Z of $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$, as an associative commutative algebra and as an $SL(2,\mathbb{Z})$ representation, is indeed extracted from a representation space of the simplest Cherednik algebra $\mathcal{H}$, defined by the relations

$$TXT = X^{-1}, \qquad TY^{-1}T = Y,$$
$$XY = \mathfrak{q}YXT^2, \quad (T-\mathfrak{q})(T+\mathfrak{q}^{-1}) = 0$$

on the generators T, X, Y, and their inverse. In these terms, the $PSL(2,\mathbb{Z})$ action is defined by the elements $\tau_+ = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\tau_- = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ being realized as the $\mathcal{H}$ automorphisms [49]

$$\tau_+ : X \mapsto X, \qquad Y \mapsto \mathfrak{q}^{-1/2}XY, \quad T \mapsto T,$$
$$\tau_- : X \mapsto \mathfrak{q}^{1/2}YX, \quad Y \mapsto Y, \qquad T \mapsto T.$$

For each $p \geqslant 3$, the authors of [50] construct a $(6p-4)$-dimensional (reducible but indecomposable) representation of $\mathcal{H}$ in which the eigensubspace of T with eigenvalue $\mathfrak{q}$ (as before, $\mathfrak{q} = e^{i\pi/p}$) is $(3p-1)$-dimensional. The associative commutative algebra structure induced on this eigensubspace in accordance with Cherednik's theory then coincides with the associative commutative algebra structure on the center Z of $\overline{\mathcal{U}}_{\mathfrak{q}}s\ell(2)$. Furthermore, the $SL(2,\mathbb{Z})$ representations constructed on this space à la Cherednik and à la Lyubashenko coincide. Also, the Radford- and Drinfeld-map images of irreducible

representations in the center can be “lifted” to the level of $\mathcal{H}$ (as eigenvectors of $X+X^{-1}$ and $Y+Y^{-1}$ respectively) [50].

6. Conclusions

Without a doubt, it would be extremely useful to rederive the results such as the equivalence of modular group representations in a more “categorical” approach; this would immediately suggest generalizations. But the quantum group “next in the queue” after $\overline{\mathfrak{U}}_{\mathfrak{q}}s\ell(2)$ and $\mathfrak{U}_{p,p'}$ is a quantum $s\ell(2|1)$ (cf. the remarks in [51]), which already requires extending many basic facts (e.g., those in [21]) to the case of quantum *super*groups.

The center of the Kazhdan–Lusztig-dual quantum group is to be regarded as the center of the corresponding logarithmic conformal field model; this calls for applications to boundary states in logarithmic models.

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Appendix A.

A.1. The center and q-characters. The center Z of a Hopf algebra U can be characterized as the set

$$\mathsf{Z}=\bigl\{y\in U\bigm|\mathrm{Ad}_x(y)=\epsilon(x)y\quad\forall x\in U\bigr\}. \tag{A.1}$$

The space of q-characters $\mathsf{Ch}=\mathsf{Ch}(U)\subset U^*$ is defined as

$$\begin{aligned}\mathsf{Ch}&=\bigl\{\beta\in U^*\bigm|\mathrm{Ad}^*_x(\beta)=\epsilon(x)\beta\quad\forall x\in U\bigr\}\\&=\bigl\{\beta\in U^*\bigm|\beta(xy)=\beta\bigl(S^2(y)x\bigr)\quad\forall x,y\in U\bigr\},\end{aligned} \tag{A.2}$$

where the coadjoint action $\mathrm{Ad}^*_a: U^*\to U^*$ is $\mathrm{Ad}^*_a(\beta)=\beta\bigl(S(a')?a''\bigr)$, $a\in U$, $\beta\in U^*$.

A.2. Quasitriangular Hopf algebras. Quasitriangular (or braided) Hopf algebras were introduced in [42] (also see [52]). A quasitriangular Hopf algebra U has an invertible element $R\in U\otimes U$ satisfying

$$\Delta^{\mathrm{op}}(x)=R\Delta(x)R^{-1}, \tag{A.3}$$

$$(\Delta\otimes\mathrm{id})(R)=R_{13}R_{23}, \tag{A.4}$$

$$(\mathrm{id}\otimes\Delta)(R)=R_{13}R_{12}, \tag{A.5}$$

the Yang–Baxter equation

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}, \tag{A.6}$$

and the relations $(\epsilon\otimes\mathrm{id})(R)=1=(\mathrm{id}\otimes\epsilon)(R)$, $(S\otimes S)(R)=R$.

A.3. Square of the antipode [40]**.** In any quasitriangular Hopf algebra, the square of the antipode is represented by a similarity transformation

$$S^2(x) = \boldsymbol{u}x\boldsymbol{u}^{-1},$$

where the *canonical element* $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \cdot\big((S \otimes \mathrm{id})R_{21}\big), \quad \boldsymbol{u}^{-1} = \cdot\big((S^{-1} \otimes S)R_{21}\big) \tag{A.7}$$

(where $\cdot(a \otimes b) = ab$) and satisfies the property

$$\Delta(\boldsymbol{u}) = M^{-1}(\boldsymbol{u} \otimes \boldsymbol{u}) = (\boldsymbol{u} \otimes \boldsymbol{u})M^{-1} \tag{A.8}$$

(where we recall that $M = R_{21}R_{12}$).

A.4. Balancing element. We also need the so-called *balancing element* $\boldsymbol{g} \in U$ that satisfies [40]

$$\begin{aligned} S^2(x) &= \boldsymbol{g}x\boldsymbol{g}^{-1} \quad \forall x \in U, \\ \Delta(\boldsymbol{g}) &= \boldsymbol{g} \otimes \boldsymbol{g}, \end{aligned} \tag{A.9}$$

The balancing element $\boldsymbol{g}$ allows constructing the "canonical" q-character associated with any (irreducible, because traces are insensitive to indecomposability) representation $\mathfrak{X}$ as the ("quantum") trace

$$\mathrm{Tr}_{\mathfrak{X}}(\boldsymbol{g}^{-1}?) \in \mathsf{Ch}(U). \tag{A.10}$$

For a Hopf algebra U with a right integral $\boldsymbol{\lambda}$, we recall the definition of a comodulus in **2.4**. Whenever a square root of the comodulus $\boldsymbol{a}$ can be calculated in U, a lemma of Drinfeld [40] states that

$$\boldsymbol{g}^2 = \boldsymbol{a}. \tag{A.11}$$

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